JOURNAL OF APPROXIMATION THEORY **88**, 154–167 (1997)
ARTICLE NO. AT963019

Multiresolution and Approximation and Hardy Spaces

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Communicated by R. DeVore

Received August 10, 1994; accepted in revised form February 27, 1996

In this paper we show that with the help of the Marcinkiewicz average of orthogonal projections on multiresolution approximation of $L^p(\mathbb{R}^n)$ for $p \leqslant 1$ built by a box spline one can construct equivalent metric in Hardy spaces. To prove this equivalence we generalize Fefferman–Stein theorem for discrete choice of parameter t. It turns out that the Marcinkiewicz average of the Ciesielski–Dürmeyer operator has similar properties as orthogonal projection. © 1997 Academic Press

1. INTRODUCTION

Following the ideas of Ciesielski [C1] in a Hardy space $H^1(T)$, we prove that Marcinkiewicz' average is also a useful tool in Hardy space $H^p(\mathbb{R}^n)$. With the help of Marcinkiewicz' average of orthogonal projections on multiresolution approximation of $L^p(\mathbb{R}^n)$, $0 < p \leqslant 1$, built by a sufficiently smooth box spline we introduce an equivalent H^p-metric in $H^p(\mathbb{R}^n)$. The crucial step in this construction is a generalization of the Fefferman–Stein theorem for discrete choice of parameter t, namely t belongs to powers of 2. Note that the convergence of orthogonal projection in H^p-metric by application of Franklin system was proved in [S] for $H^p(\mathbb{R}^n)$ and for $H^p(T^n)$ in [W]. In fact, it was proved that the Franklin system forms an unconditional basis in Hardy spaces. The rates of convergence were treated in [O].

First we recall the definition and properties of a box spline. Let V be a family of vectors from $\mathbb{Z}^n \backslash 0$,

$$V = v_1, ..., v_s$$

such that

$$\text{span}\{V\} = \mathbb{R}^n.$$

0021-9045/97 $25.00

Throughout this paper it is assumed that V is unimodular, i.e.,

$$\forall_{X \subset V} \qquad \# X = n \qquad \text{then} \quad |\det X| \leqslant 1, \tag{1.1}$$

where $\# X$ denotes the cardinality of X. A box spline associated with V is then defined as a function for which the relation

$$\int_{\mathbb{R}^n} f(x)\, B(x \mid V)\, dx = \int_{[0,1]^s} f\left(\sum_{i=1}^{s} u_i v_i\right) du \tag{1.2}$$

holds for all continuous f on $\mathbb{R}^n$.

Let

$$\varrho_V = \max\{r \colon \forall_{X \subset V}\ \# X = r,\ \operatorname{span}\{V \backslash X\} = \mathbb{R}^n\}.$$

It is known that

$$B(\cdot \mid V) \in C^{\varrho_V - 1} - C^{\varrho_V}.$$

Let V_0 be the closed subspace of $L^2(\mathbb{R}^n)$ spanned by integer translates of the box spline $B(\cdot \mid V)$, i.e.,

$$V_0 = \operatorname{span}_{L^2}\{B(\cdot - \alpha \mid V) \colon \alpha \in \mathbb{Z}^n\} \tag{1.3}$$

and let us introduce, for $0 < p \leqslant 1$, the closed subspace of $L^p(\mathbb{R}^n)$

$$V_0^p = \operatorname{span}_{L^p}\{B(\cdot - \alpha \mid V) \colon \alpha \in \mathbb{Z}^n\}. \tag{1.4}$$

It is known that the assumption that V is unimodular implies that the integer translates of the box spline $B(\cdot - \alpha \mid V) \colon \alpha \in \mathbb{Z}^n$ constitute a Riesz basis in V_0.

Introduce the scaling operator σ and the shift operator τ:

$$\sigma_\eta f = f(\eta\, \cdot) \qquad \text{for} \quad \eta \in \mathbb{R}$$

and respectively

$$\tau_t f(x) = f(x - t) \qquad \text{for} \quad t \in \mathbb{R}$$

Let

$$V_j = \sigma_{2^j} V_0, \qquad V_j^p = \sigma_{2^j} V_0^p, \qquad j \in \mathbb{Z}.$$

From the (50) Theorem in Section 5 of [BHR] we have that the sequence $(V_j)_{j \in Z}$ of closed subspaces of $L^2(\mathbb{R}^n)$ forms *a multiresolution approximation* of $L^2(\mathbb{R}^n)$, i.e.,

(i) $V_j \subset V_{j+1}$ for $j \in \mathbb{Z}$,

(ii) $f \in V_j \Rightarrow f(\cdot - 2^{-j}\alpha) \in V_j$ for $j \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}^n$,

(iii) $f \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1}$,

(iv) there is linear isomorphism from l^2 onto V_0 which commutes with the shift operators τ_α, $\alpha \in \mathbb{Z}^n$,

(v) $\bigcap_{j \in \mathbb{Z}} V_j = 0$,

(vi) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^n)$.

Let us note that respectively

$(V_j^p)_{j \in Z}$ forms the multiresolution approximation of $L^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$.

Let us outline the proof in the case of (iv). Since $0 < p \leqslant 1$, we see that

$$\int_{\mathbb{R}^n} \left| \sum_{\alpha \in \mathbb{Z}^n} a_\alpha B(x - \alpha \mid V) \right|^p dx \leqslant \int_{\mathbb{R}^n} \sum_{\alpha \in \mathbb{Z}^n} |a_\alpha B(x - \alpha \mid V)|^p \, dx$$
$$\leqslant C_1 \sum_{\alpha \in \mathbb{Z}^n} |a_\alpha|^p,$$

where

$$C_1 = \int B(x \mid V)^p \, dx.$$

On the other hand, V is unimodular hence the sequence of functions

$$\{B(\cdot - \alpha \mid V)\}_{\alpha \in \mathbb{Z}^n}$$

is locally linearly independent, see [BHR, (57) Theorem]. In particular, all integer shifts of $B(\cdot \mid V)$ having support with non-void intersection with the cube $[0, 1]^n$ are linearly independent. From this we conclude that there are constants $C_2 > 0$, $\eta \in \mathbb{R}^n$ and $N > 0$ such that

$$C_2 \sum_{|\alpha - \eta| < N} |a_\alpha|^p \leqslant \int_{[0,1]^n} \left| \sum_{\alpha \in \mathbb{Z}^n} a_\alpha B(x - \alpha \mid V) \right|^p dx.$$

Thus for all $\beta \in \mathbb{Z}^n$,

$$C_2 \sum_{|\alpha - \eta - \beta| < N} |a_\alpha|^p \leqslant \int_{[0,1]^n + \beta} \left| \sum_{\alpha \in \mathbb{Z}^n} a_\alpha B(x - \alpha \mid V) \right|^p dx.$$

Finally,

$$(2N)^n C_2 \sum_{\alpha \in \mathbb{Z}^n} |a_\alpha|^p \leqslant \int_{\mathbb{R}^n} \left| \sum_{\alpha \in \mathbb{Z}^n} a_\alpha B(x - \alpha \mid V) \right|^p dx. \quad \blacksquare$$

Let us introduce the orthogonal projections onto multiresolution analysis

$$P_j\colon L^2(\mathbb{R}^n) \to V_j \tag{1.5}$$

for $j \in \mathbb{Z}$. Note that

$$P_j = \sigma_{2^{-j}} P_0(\sigma_{2^j} f).$$

If ϱ_v is large, by applying the formula for P_j it is easy to check that these operators, considered as acting from the Hardy space $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$,

$$P_j\colon H^p(R^n) \to V_0^p = \operatorname{span}_{L^p}\{B(\cdot - \alpha \mid V)\colon \alpha \in \mathbb{Z}^n\}$$

are continuous. As usual, $[x]$ denotes the integer part of x.

The main result of this paper is the following box spline maximal function characterization of $H^p(\mathbb{R}^n)$.

Main Theorem 1.6. *Let $0 < p \leqslant 1$ and $\varrho_V - 1 \geqslant 3[n/p]$, then there are constants $C_1, C_2 > 0$ depending on p such that for all distribution f from the Hardy space $H^p(\mathbb{R}^n)$*

$$C_1 \|f\|_{H^p} \leqslant \left\| \sup_{j \in \mathbb{Z}} \left| \int_{[0,1)^n} \tau_{-t} P_j(\tau_t f)(\cdot)\, dt \right| \right\|_{L^p(\mathbb{R}^n)} \leqslant C_2 \|f\|_{H^p}.$$

This theorem follows from Theorems 2.2, 3.9, and 3.10 in the paper.

2. HARDY SPACES

The following theorem generalizes the well known Fefferman–Stein results. For convenience introduce for $t > 0$

$$L_t(x) = 1/t^n L(x/t).$$

A function L (resp., a sequence $\{C_\alpha\}$) decays exponentially if there are constants $C > 0$ and $0 < q < 1$ such that

$$|L(x)| \leqslant C q^{|x|}, \qquad x \in \mathbb{R}^n,$$

and resp.,

$$|C_\alpha| \leqslant C q^{|\alpha|}, \qquad \alpha \in Z^n.$$

The function L is *refinable* if for all $m \in N \backslash 0$ there is a mask $\{C_\alpha^m\}_{\alpha \in \mathbb{Z}^n}$ decaying exponentially such that

$$L(x) = \sum_{\alpha \in \mathbb{Z}^n} C_\alpha^m L(mx - \alpha). \tag{2.1}$$

THEOREM 2.2. *Assume that there is a function $L \in C^{6[n/p]}(\mathbb{R}^n)$ which decays exponentially with all its admissible derivatives, is refinable, and for which*

$$\hat{L}(0) \neq 0.$$

Then there are constants $C_1, C_2 > 0$ depending on p such that

$$C_1 \|f\|_{H^p} \leqslant \|\sup_{j \in \mathbb{Z}} |L_{2^j} * f|\|_{L^p(\mathbb{R}^n)} \leqslant C_2 \|f\|_{H^p}.$$

From [FS] we infer that quasi-norm in Hardy spaces may be defined by a function $\varphi \in C^k(\mathbb{R}^n)$, where k is large enough (for example, $k \geqslant 3[n/p]$), decaying exponentially with its admissible derivatives,

$$\hat{\varphi}(0) \neq 0,$$

by formula

$$\|f\|_{H^p} = \|\sup_{|\cdot - y| < t} |\varphi_t * f(y)|\|_{L^p(\mathbb{R}^n)} \tag{2.3}$$

for all $f \in H^p$. This metric is equivalent with

$$\|f\|_{H^p} \sim \|\sup_{0 < t} |\varphi_t * f|\|_{L^p(\mathbb{R}^n)}. \tag{2.4}$$

Our task is to change continuous parameter $t \in \mathbb{R}$ into discrete

$$t \in \{2^j : j \in \mathbb{Z}\} \tag{2.5}$$

if we assume that φ is refinable.

Proof of Theorem 2.2. Every interval

$$[1/2^{m+1}, 1/2^m] \qquad m \in \mathbb{Z}$$

we divide into 2^k equal parts:

$$1/2^{m+1} = t_0^m < t_1^m < \cdots < t_{2^k}^m = 1/2^m$$

and

$$t_j^m = \frac{2^k + j}{2^{m+k+1}} \qquad j = 0, 1, ..., 2^k.$$

Let

$$W_k = \bigcup_m \{ t_0^m, ..., t_{2^k}^m \}.$$

We choose k later. We show that

$$\|f\|_{H^p} \sim \| \sup_{t \in W_k} |L_t * f| \|_{L^p(\mathbb{R}^n)}.$$

On the one side it is obvious. Fix $x \in \mathbb{R}^n$, $f \in H^p(\mathbb{R}^n)$ and set

$$h_x(t) = L_t * f(x).$$

Then

$$h_x'(t) = (-n/t)\, h_x(t) - (1/t)\, K_t * f(x),$$

where

$$K(y) = \sum_{i=1}^n y_i \frac{\partial L}{\partial x_i}(y).$$

Lagrange's Theorem implies that for

$$t \in [t_j^m, t_{j+1}^m]$$

we have

$$|h_x(t) - h_x(t_j^m)| \leqslant 1/2^{k+m+1} \sup_{s \in [t_j^m, t_{j+1}^m]} |h_x'(s)|.$$

Consequently

$$|h_x(t) - h_x(t_j^m)| \leqslant n/2^k \sup_{s \in [t_j^m, t_{j+1}^m]} |h_x(s)| + 1/2^k \sup_{0<t} |K_t * f(x)|.$$

Taking supremum we get

$$\sup_{s \in [t_j^m, t_{j+1}^m]} |h_x(s)| \leqslant |h_x(t_j^m)| + n/2^k \sup_{s \in [t_j^m, t_{j+1}^m]} |h_x(s)| + 1/2^k \sup_{0<t} |K_t * f(x)|.$$

Hence

$$(1 - n/2^k) \sup_{s \in [t_j^m, t_{j+1}^m]} |h_x(s)| \leqslant \sup_{t \in W_k} |h_x(t)| + 1/2^k \sup_{0<t} |K_t * f(x)|.$$

Then

$$(1 - n/2^k) \sup_{s>0} |L_s * f(x)| \leqslant \sup_{t \in W_k} |L_t * f(x)| + 1/2^k \sup_{0<t} |K_t * f(x)|.$$

Integrating over $\mathbb{R}^n$ we obtain

$$(1-n/2^k)\left(\int_{\mathbb{R}^n}\sup_{0<t}|L_t*f(x)|^p\,dx\right)^{1/p}$$
$$=2^{1/p-1}\left(\left(\int_{\mathbb{R}^n}\sup_{t\in W_k}|L_t*f(x)|^p\,dx\right)^{1/p}+1/2^k\left(\int_{\mathbb{R}^n}\sup_{0<t}|K_t*f(x)|^p\,dx\right)^{1/p}\right).$$

From the Fefferman–Stein Theorem we infer that there is a constant $C(L,K)$ such that

$$\|\sup_{0<t}|K_t*f|\|_{L^p(\mathbb{R}^n)}\leqslant C(L,K)\,\|\sup_{0<t}|L_t*f|\|_{L^p(\mathbb{R}^n)}.$$

Consequently

$$\left(1-\frac{n+2^{1/p-1}C(L,K)}{2^k}\right)\|\sup_{0<t}|L_t*f(x)|\|_{L^p(\mathbb{R}^n)}$$
$$\leqslant 2^{1/p-1}\,\|\sup_{t\in W_k}|L_t*f(x)|\|_{L^p(\mathbb{R}^n)}.$$

Now it is clear that we choose k such that

$$n+2^{1/p-1}C(L,K)<2^k.$$

Now it is sufficient to prove that

$$\|\sup_{j\in\mathbb{Z}}|L_{2^j}*f|\|_{L^p(\mathbb{R}^n)}\sim\|\sup_{t\in W_k}|L_t*f|\|_{L^p(\mathbb{R}^n)}.$$

Let $t\in W_k$, then

$$t=(2^k+q)/2^{k+i}$$

for certain $1\leqslant q\leqslant 2^k$. Let $m=2^k+q$. From (2.1) we see that

$$|L_t*f(x)|\leqslant\sum_{\alpha\in\mathbb{Z}^n}|C_\alpha^m/t^nL(m\cdot/t-\alpha)*f(x)|$$
$$\leqslant\sum_{\alpha\in\mathbb{Z}^n}|C_\alpha^m/t^nL(m/t(\cdot-t\alpha/m))*f(x)|$$
$$\leqslant\sum_{\alpha\in\mathbb{Z}^n}|C_\alpha^m/m^nL_{t/m}(\cdot-t\alpha/m)*f(x)|.$$

Note that

$$t/m\in\{2^j\colon j\in\mathbb{Z}\}.$$

From this and the fact $0<p\leqslant 1$, denoting $C_\alpha=\max_m |C_\alpha^m$, it follows that

$$\sup_{t\in W_k} |L_t * f(x)|^p \leqslant \sum_{\alpha\in\mathbb{Z}^n} |C_\alpha\cdot 2^{-nk}|^p \sup_{j\in\mathbb{Z}} |L_{2^j}((\cdot-2^j\alpha))*f(x)|^p.$$

Integrating both sides we get

$$\int_{\mathbb{R}^n}\sup_{t\in W_k} |L_t * f(x)|^p\,dx \leqslant \sum_{\alpha\in\mathbb{Z}^n} |C_\alpha\cdot 2^{-nk}|^p \int_{\mathbb{R}^n}\sup_{j\in\mathbb{Z}} |L_{2^j}*f(x-2^j\alpha)|^p\,dx.$$

$$\leqslant \sum_{\alpha\in\mathbb{Z}^n} |C_\alpha\cdot 2^{-nk}|^p \int_{\mathbb{R}^n}\sup_{j\in\mathbb{Z}}\sup_{|y-x|\leqslant 2^j|\alpha|} |L_{2^j}*f(y)|^p\,dx.$$

From Lemma 1 [FS pg. 166] with the obvious changes we obtain

$$\int_{\mathbb{R}^n}\sup_{t\in W_k} |L_t * f(x)|^p\,dx$$

$$\leqslant C\sum_{\alpha\in\mathbb{Z}^n} |C_\alpha\cdot 2^{-nk}|^p\,|\alpha|^{n/p} \int_{\mathbb{R}^n}\sup_{j\in\mathbb{Z}}\sup_{|y-x|<2^j} |L_{2^j}*f(y)|^p\,dx,$$

C_α^m decays exponentially. Therefore C_α decays that way, too. Consequently

$$\int_{\mathbb{R}^n}\sup_{t\in W_k} |L_t * f(x)|^p\,dx \leqslant C\int_{\mathbb{R}^n}\sup_{j\in\mathbb{Z}}\sup_{|y-x|<2^j} |L_{2^j}*f(y)|^p\,dx.$$

From the Fefferman–Stein Theorem 11 (page 183) with discrete choice of t we conclude that

$$\int_{\mathbb{R}^n}\sup_{t\in W_k} |L_t * f(x)|^p\,dx \leqslant C\int_{\mathbb{R}^n}\sup_{j\in\mathbb{Z}} |L_{2^j}*f(x)|^p\,dx$$

which finishes the proof. ∎

3. MARCINKIEWICZ' AVERAGE

One of the known properties of the box splines is the following fact.

THEOREM 3.1 [DDL]. *Let $\varrho_V\geqslant 1$. Then for each $m\in N\backslash 0$ there is a finite sequence of coefficients (b_α^m) such that*

$$B(x\mid V)=\sum_{\alpha\in\mathbb{Z}^n} b_\alpha^m B(mx-\alpha\mid V).$$

Let us introduce a trigonometric polynomial P

$$P(x)=\sum_{\alpha\in\mathbb{Z}^n} B(\alpha \mid Y)\, e^{2\pi i\alpha\cdot x}, \tag{3.2}$$

where a family

$$Y=\{V, -V\}.$$

THEOREM 3.3 [BHR IV (28)]. *The family V is unimodular if and only if*

$$\forall_{x\in\mathbb{R}^n}\, P(x)\neq 0.$$

The periodic function $G=1/P$ has Fourier expansion

$$G(x)=\sum_{\alpha\in\mathbb{Z}^n} g_\alpha e^{2\pi i\alpha\cdot x}, \tag{3.4}$$

where coefficients decay exponentially [see JM]. Let

$$B^*(x)=\sum_{\alpha\in\mathbb{Z}^n} g_\alpha B(\cdot-\alpha \mid V). \tag{3.5}$$

From construction of the sequence (g_α) we conclude that B^* is the biorthogonal function, i.e.

$$B^*\in V_0$$

and

$$\int_{\mathbb{R}^n} B(x-\alpha \mid V)\, B^*(x)\, dx=\delta_{0,\alpha},$$

where δ denotes the Kroneker symbol.

The *fundamental function* Θ is given by

$$\Theta(x)=\int_{\mathbb{R}^n} B(x+y \mid V)\, B^*(y)\, dy. \tag{3.6}$$

From (3.4) and (3.5), it follows that

$$\Theta(x)=\sum_{\alpha\in Z^n} g_\alpha B(x-\alpha \mid Y) \tag{3.7}$$

and

$$B(x \mid Y)=\sum_{\alpha\in Z^n} B(\alpha \mid Y)\, \Theta(x-\alpha). \tag{3.8}$$

THEOREM 3.9. *Let* $0<p\leqslant 1$ *and* $\varrho_V-1\geqslant 3[n/p]$. *Then Marcinkiewicz' average of the orthogonal projections is represented by the convolution with the fundamental function, i.e., for all* $f\in H^p(\mathbb{R}^n)$,

$$\int_{[0,1)^n} \tau_{-t} P_j(\tau_t f)(x)\,dt=(2^j)^n\,(\sigma_{2^j}\Theta)*f(x)=\Theta_{2^{-j}}*f(x).$$

Proof. Put

$$(f,g)=\int_{\mathbb{R}^n} f(x)\,g(x)\,dx.$$

Let $f\in L^2(\mathbb{R}^n)$. Then

$$P_j f(x)=\sum_{\alpha\in Z^n}(2^j)^n\,(f,B^*(2^j\cdot-\alpha))\,B(2^jx-\alpha\mid V).$$

Hence

$$\begin{aligned}
&\int_{[0,1)^n}(P_j(\tau_t f))(x+t)\,dt\\
&\quad=\int_{[0,1)^n}\sum_{\alpha\in Z^n}2^{jn}(\tau_t f,B^*(2^j\cdot-\alpha))\,B(2^j(x+t)-\alpha\mid V)\,dt\\
&\quad=\sum_{\alpha\in Z^n}\int_{[0,1)^n}2^{jn}(f,B^*(2^j(\cdot+t-\alpha/2^j)))\,B(2^j(x+t-\alpha/2^j)\mid V)\,dt\\
&\quad=(2^{jn})^2\int_{\mathbb{R}^n}(f,B^*(2^j(\cdot+t)))\,B(2^j(x+t)\mid V)\,dt\\
&\quad=(2^{jn})^2\int_{\mathbb{R}^n}\int_{R^n}f(u)\,B^*(2^j(u+t))\,B(2^j(x+t))\mid V)\,dt\,du\\
&\quad=2^{jn}\int_{\mathbb{R}^n}f(u)\,\Theta(2^jx-2^ju)\,du=2^{jn}f*(\sigma_{2^j}\Theta)(x).
\end{aligned}$$

Since the functions $f\in H^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ are dense in $H^p(\mathbb{R}^n)$, this completes the proof. ∎

From (3.7), (3.8), and Theorem 3.1 we have following theorem.

THEOREM 3.10. *For all* $m\in N\backslash 0$,

$$\Theta(x)=\sum_{\alpha}\theta_\alpha^m\,\Theta(mx-\alpha),$$

where the mask decays exponentially, i.e.,

$$|\theta_\alpha^m| \leqslant C_m q^{|\alpha|} \qquad \alpha \in \mathbb{Z}^n,$$

for constants $C_m > 0$ *and* $0 < q < 1$. *Constant* C_m *depends on m. Moreover,*

$$\hat{\Theta}(0) \neq 0.$$

Remarks. Let us introduce a discrete convolution of given sequences $a = \{a_\alpha\}$ and $b = \{b_\alpha\}$. This is a sequence $a * b$ such that

$$(a * b)_\beta = \sum_{\alpha \in \mathbb{Z}^n} a_{\beta - \alpha} b_\alpha. \tag{3.11}$$

Put

$$a^k = \underbrace{a * \cdots * a}_{k} \qquad k \geqslant 1 \tag{3.12}$$

and

$$\delta = \{\delta_{0,\alpha}\}. \tag{3.13}$$

Let us introduce a class of Ciesielski–Dürmeyer operators. Let $r^\rho \in N$ be a sequence such that

$$r^\rho = \delta + M + M^2 + \cdots + M^\rho,$$

where a sequence M is given by

$$M = \delta - \{B(\alpha \mid Y)\}_{\alpha \in \mathbb{Z}^n}$$

and Y is the family $\{V, -V\}$.

The Ciesielski–Dürmeyer operator (quasi-projection) associated with a family V and sequence r^ρ is given by the formula (see [C2])

$$Q^{V,V,\rho}(f) = \sum_{\alpha \in \mathbb{Z}^n} (f, B(\cdot - \alpha \mid V) * r^\rho)\, B(\cdot - \alpha \mid V),$$

where

$$B(\cdot - \alpha \mid V) * r^\rho = \sum_{\alpha \in \mathbb{Z}^n} r_\alpha^\rho B(\cdot - \alpha \mid V).$$

From a paper [CDR] we infer that when $\rho \to \infty$ then

$$r^\rho \to g$$

the sequence g being given in (3.4) and

$$Q^{V,V,\rho}(f) \to P_0(f).$$

It is interesting that the properties of projections P_j described by Theorem 1.5 inherit the operators

$$Q_j^{V,V,\rho} = \sigma_{2^{-j}} Q^{V,V,\rho}(\sigma_{2^j} f)$$

for $\rho = 0$ and for large ρ.

4. APPLICATION

We can obtain a nice application of Theorem 1.6 for $H^1(\mathbb{R}^n)$. Namely,

THEOREM 4.1. *Let $\varrho_V \geqslant 3n$. Then there are constants $C_1, C_2 > 0$ such that for all functions f from the Hardy space $H^1(\mathbb{R}^n)$*

$$C_1 \|f\|_{H^1} \leqslant \int_{[0,1)^n} \|\sup_{j \in \mathbb{Z}} |P_j(\tau_t f)(\cdot)|\|_{L^1(\mathbb{R}^n)} \, dt \leqslant C_2 \|f\|_{H^1}.$$

Proof. The left side of inequality is obvious from Theorem 1.6 since

$$\sup_{j \in \mathbb{Z}} \left| \int_{[0,1)^n} \tau_{-t} P_j \tau_t f \, dt \right| \leqslant \int_{[0,1)^n} \sup_{j \in \mathbb{Z}} |\tau_{-t} P_j \tau_t f| \, dt.$$

To prove the right side of the inequality we recall some properties of box splines. If

$$c_V = \sum_{j=1}^{s} v_j$$

then

$$B(x \mid V) = B(c_V - x \mid V) \tag{4.2}$$

From (3.4), (3.5) we obtain that

$$B(\cdot \mid V) = \sum_{\alpha \in Z^n} a_\alpha B^*(\cdot - \alpha \mid V),$$

where $a_\alpha = B(\alpha \mid Y)$. Hence by applying (3.5), (4.2), and fact that $a_\alpha = a_{-\alpha}$ we get

$$B^*(x) = B^*(c_V - x). \tag{4.3}$$

By definition

$$P_0 f = \sum_{\alpha \in Z^n} (f, B^*(\cdot - \alpha))\, B(\cdot - \alpha \mid V).$$

Since $B(\cdot \mid V)$ is nonnegative function with compact support and

$$\sum_{\alpha \in Z^n} B(x - \alpha \mid V) = 1$$

there is an $N > 0$ such that

$$\begin{aligned} |P_j f(x)| &\leqslant \left| \sum_{\alpha \in Z^n} (\sigma_{2^j} f, B^*(\cdot - \alpha))\, B(2^{-j} x - \alpha \mid V) \right| \\ &\leqslant \sum_{|x - 2^j \alpha| < N 2^j} |(\sigma_{2^j} f, B^*(\cdot - \alpha))|. \end{aligned}$$

From (4.3) we get

$$|P_j f(x)| \leqslant \sum_{|x - 2^j \alpha| < N 2^j} |(f * B^*_{2_j}(2^j \alpha + 2^j c_V))|.$$

Hence

$$\begin{aligned} &\|\sup_{j \in \mathbb{Z}} |P_j f| \, \|_{L^1(\mathbb{R}^n)} \\ &\quad \leqslant N^n \, \|\sup_{j \in \mathbb{Z}} \{ f * (B^*)_{2^j}(y) : |x - y| < (N + |c_V|)\, 2^j \} \|_{L^1(\mathbb{R}^n)} \leqslant C \, \|f\|_{H^1}. \end{aligned}$$

Since the norm in Hardy space H^1 is invariant under translation this completes the proof. ▮

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